

Chapter 5

Structural Equation Models with Latent Variables

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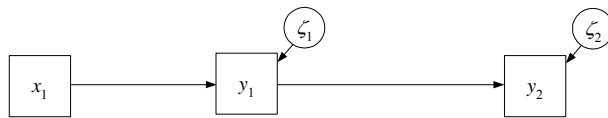
5.1 Introduction to Structural Equation Models

Objectives

- ▶ Introduce SEM as a flexible combination of simultaneous equation modeling (path analysis) and CFA

Simultaneous Equation Models

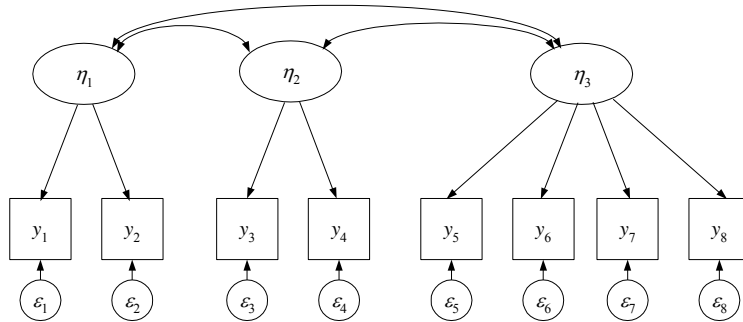
- ▶ With simultaneous equation models (path analysis) we could fit complex structural models for observed variables



- ▶ Here, we have a relatively simple simultaneous equation model where the effect of x is mediated by y_1 .
- ▶ Although widely used, we have shown that fitting this model will almost certainly result in biased effect estimates due to failure to account for measurement error in the observed variables

Confirmatory Factor Analysis

- ▶ In factor analysis, we inferred the presence of underlying, error-free latent variables from correlated observed variables

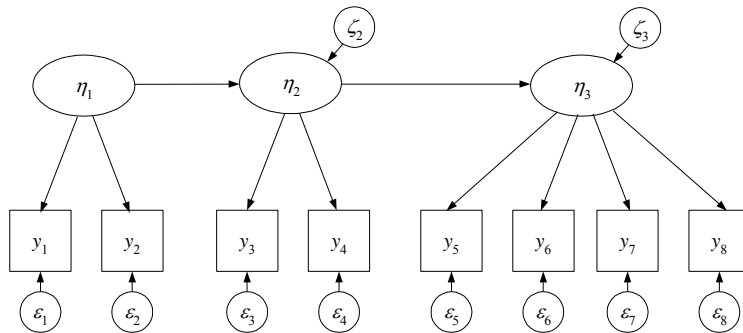


- ▶ Yet here we have not tested any structural model of interest

Here the covariance matrix among the factors is saturated, including non-directional associations among all latent variables.

Structural Equation Modeling

- ▶ In structural equation models, we combine the structural model of path analysis with the measurement model of CFA

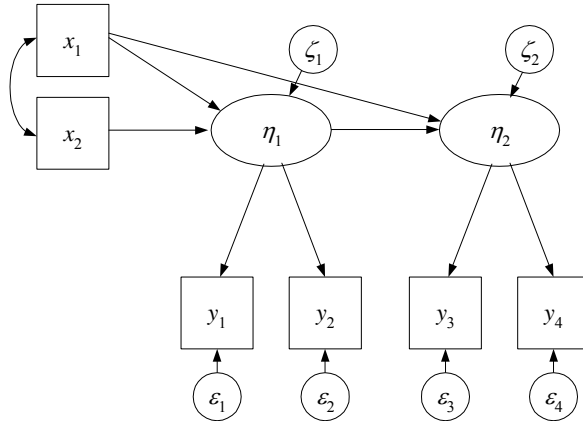


- ▶ We are now estimating structural relations among latent variables that are unbiased by measurement error

Here a causal structure has been applied to the covariance matrix among the factors

Fixed- x , Exogenous predictors

- ▶ We can also include manifest fixed regressors, or exogenous predictors, in our models, as in the model below



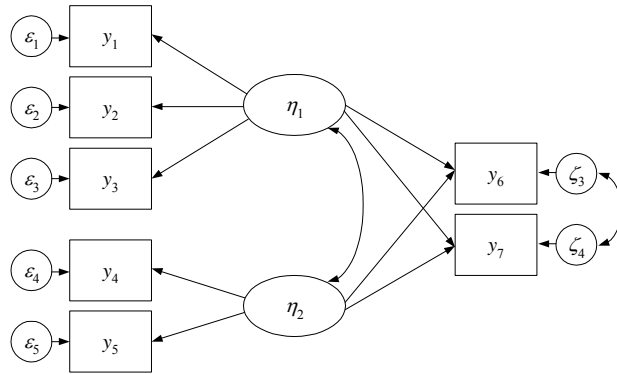
Fixed- x , Exogenous predictors

- ▶ As with the simultaneous equation model, we make no distributional assumptions about the exogenous predictors
- ▶ Exogenous predictors can even be binary
 - ▶ Gender
 - ▶ Child of an Alcoholic
- ▶ As in simultaneous equation models, however, exogenous predictors are still assumed to be measured without error. If this is not true, effect estimates may be biased.
 - ▶ Preferable to use multiple indicator latent factors as predictors whenever measurement error is likely

As with path analysis / simultaneous equation models, we can include nominal exogenous predictors via coding variables.

Models with Manifest Outcomes

- ▶ In some models, we may also have manifest variables as outcome variables, as in the model shown below.

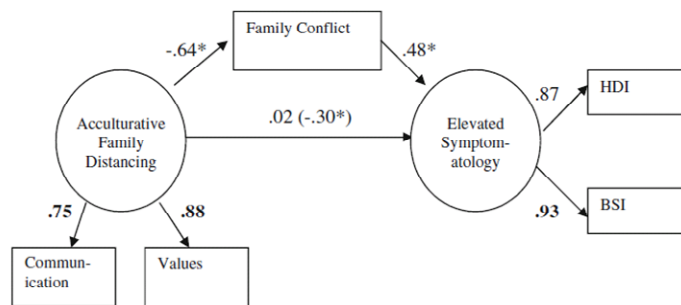


Models with Manifest Outcomes

- ▶ Like simultaneous equation models, manifest outcomes are assumed to be free of measurement error
 - ▶ Unlike measurement error in predictors, measurement error in outcomes does not bias the structural paths leading to the outcome.
 - ▶ It does, however, cause under-estimation of the amount of variance explained in the outcome, and inflation of the standard error of the structural paths leading to the outcome (resulting in lower power).
 - ▶ Thus, given sufficient sample size, still best to use full SEM including multiple indicator latent outcomes instead of manifest outcomes when measurement error is likely.

Example Applications

- ▶ Hwang & Woods (2009) used a relatively simple SEM to show that the mental health status of Asian and Latino American college students is predicted by the acculturation gap with their parents, as mediated by family conflict.

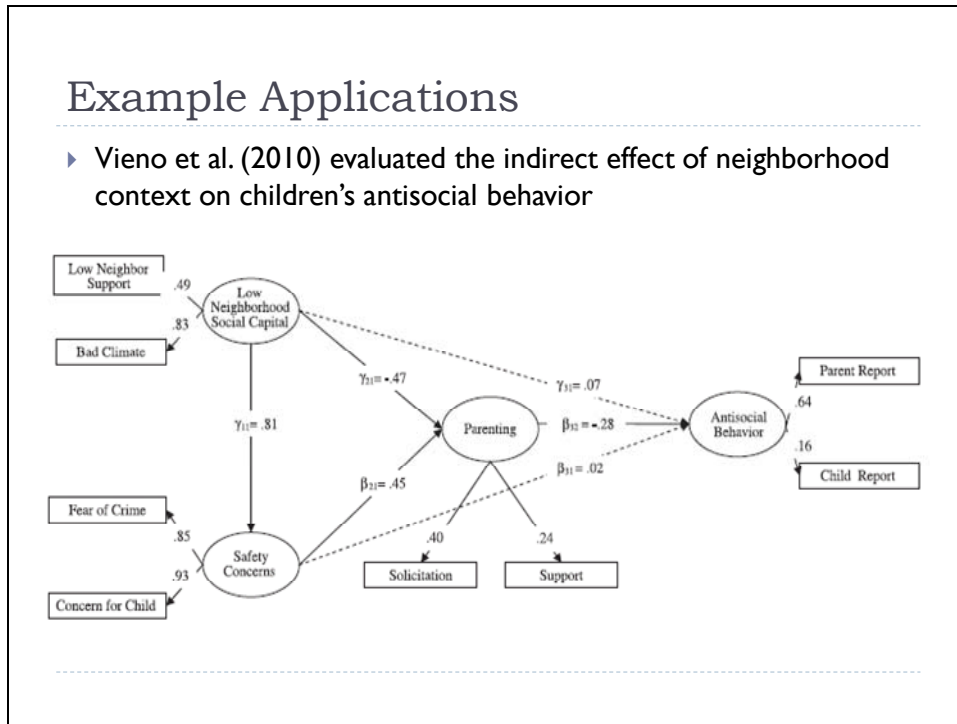


Hwang W.-C. & Wood, J.J. (2009). Acculturative family distancing: links with self-reported symptomatology among Asian Americans and Latinos. *Child Psychiatry and Human Development*, 40, 123–138.

Abstract from manuscript: *Objective* Our knowledge of how acculturative processes affect families remains quite limited. This article tests whether acculturative family distancing (AFD), a more proximal and problem-oriented measure of the acculturation gap, influences the mental health status of Asian American and Latino college students. AFD occurs along two dimensions: communication difficulties and cultural value incongruence. *Methods* Data were collected from 186 Asian American ($n = 107$) and Latino ($n = 79$) undergraduates, who provided self-reports on psychological problems, depressive symptoms, and family conflict. A new self-report measure of AFD evidencing good psychometric properties was used to test hypothesized relations among these variables in structural equation models (SEM). *Results* For both Asian American and Latinos, results indicated that higher levels of AFD were associated with higher psychological distress and greater risk for clinical depression, and that family conflict mediated this relation. *Conclusion* AFD processes were associated with the mental health of students and the functioning of their families. These findings highlight potential foci to address in prevention and intervention programs, such as improving communication and teaching families how to negotiate cultural value differences.

Example Applications

- ▶ Vieno et al. (2010) evaluated the indirect effect of neighborhood context on children's antisocial behavior

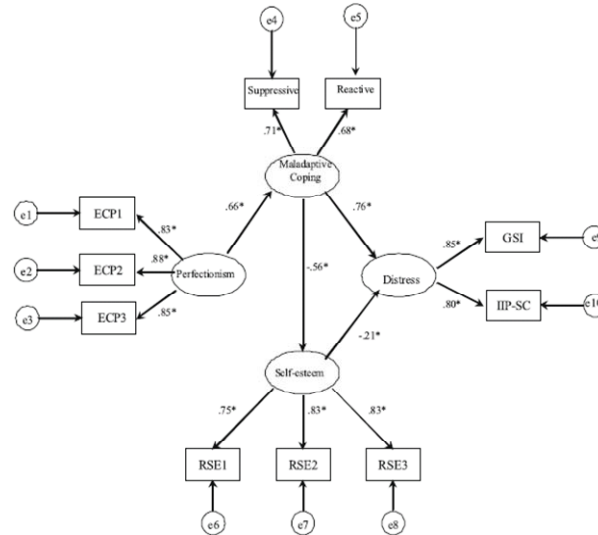


Vieno, A., Nation, M., Perkins, D.D., Pastore, M. & Santinello, M. (2010). Social capital, safety concerns, parenting, and early adolescents' antisocial behavior. *Journal of Community Psychology, 38*, 314-328.

Abstract from manuscript: This study explores the relations between neighborhood social capital (neighbor support and social climate), safety concerns (fear of crime and concern for one's child), parenting (solicitation and support), and adolescent antisocial behavior in a sample of 952 parents (742 mothers) and 588 boys and 559 girls from five middle schools (sixth through eighth grades) in a midsize Italian city. In structural equation models, social capital is strongly and inversely related to safety concerns and positively related to parental support and solicitation. In turn, safety concerns are also positively related to parental support and solicitation. Social capital and safety concerns have indirect effects on children's antisocial behavior through their effects on parenting. Implications are discussed for parenting and community-based interventions to prevent or reduce youth antisocial behaviors

Example Applications

- ▶ Park, Heppner & Lee (2010) evaluated whether maladaptive coping and self-esteem mediate the relationship between perfectionism and distress



Park, H.-J., Heppner, P.P. & Lee, D.-G. (2010). Maladaptive coping and self-esteem as mediators between perfectionism and psychological distress. *Personality and Individual Differences* 48, 469–474.

Abstract from manuscript: This study with 508 Korean college students examined the mediation effects of maladaptive coping styles and self-esteem on the links of evaluative concerns perfectionism and psychological distress. Structural equation modeling analyses supported a full mediation effect of maladaptive coping between evaluative concerns perfectionism and distress. The final model also revealed a significant path from evaluative concerns perfectionism through maladaptive coping and self-esteem to distress. Furthermore, a multi-group analysis found that male college students with evaluative concerns perfectionism tend to use maladaptive coping strategies more compared to their female counterparts. The findings provided not only external validity for the full mediation effect of coping but also evidence of more complex relations among the variables.

Summary

- ▶ Structural equation models represent the natural combination of simultaneous equation models with confirmatory factor analysis
 - ▶ The full model is very flexible, permitting complex structural models involving both observed and latent predictors and outcomes
 - ▶ The presence of the measurement model permits the estimation of effects without bias or standard error inflation due to measurement error
 - ▶ We will now see that fitting SEMs is also a straightforward extension of procedures we have covered for simultaneous equation models and CFA
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5.2 Fitting Structural Equation Models

Objectives

- ▶ Show specification of SEMs in path diagram and matrix form
 - ▶ Describe model identification rules
 - ▶ Describe process of model estimation, evaluation and re-specification
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Fitting SEMs

- ▶ As with simultaneous equation models and CFA, model fitting involves the following steps
 - ▶ Specification
 - ▶ Identification
 - ▶ Estimation
 - ▶ Evaluation
 - ▶ Potential re-specification
 - ▶ Interpretation
 - ▶ Fortunately, we can draw on the knowledge base we have built for simultaneous equation models and CFA to develop general principles for the full structural equation model with latent variables.
-

Model Specification

- ▶ As with prior models, the initial specification of the model should be heavily guided by theory.
 - ▶ Which variables are predictors, mediators, and outcomes?
 - ▶ Which variables have causal effects, and which are merely associated?
 - ▶ The model can be specified graphically via a path diagram, or in equations using an expanded set of model matrices.
 - ▶ Our model matrices now combine matrices used previously with simultaneous equation models and matrices used with CFA.
 - ▶ The full SEM consists of two parts:
 - ▶ The measurement model (like CFA)
 - ▶ The structural model (like path analysis)
-

Model Specification

- ▶ The *measurement model* is used to define the relationships of the indicator variables to the latent variables:

$$\mathbf{y}_i = \mathbf{v} + \mathbf{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i, \text{ where } \text{VAR}(\boldsymbol{\varepsilon}_i) = \boldsymbol{\Theta}$$

This is identical to the measurement model of CFA

Model Specification

- ▶ The *structural model* is used to define the relationships among the latent variables and between the latent variables and any fixed- x exogenous predictors:

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta}_i + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\zeta}_i, \text{ where } \text{VAR}(\boldsymbol{\zeta}_i) = \boldsymbol{\Psi}$$

This is identical to the structural model from path analysis, but latent variables have replaced observed variables as outcomes

- ▶ Compare to

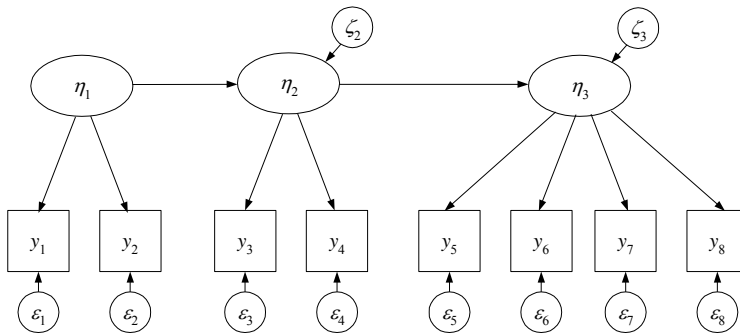
Regression: $y_i = \alpha + \boldsymbol{\gamma}'\mathbf{x}_i + \zeta_i$

Path Analysis: $\mathbf{y}_i = \boldsymbol{\alpha} + \mathbf{B}\mathbf{y}_i + \boldsymbol{\Gamma}\mathbf{x}_i + \boldsymbol{\zeta}_i$

Notice that the structural model is the same as path analysis, except that the latent variables defined by the measurement model have replaced the observed variables.

Model Specification: Example

- ▶ Consider specification of the following model



Model Specification: Example

- ▶ Using scaling indicators to set the metric of the latent variables, the *measurement model* is

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \\ y_{6i} \\ y_{7i} \\ y_{8i} \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \\ 0 \\ v_4 \\ 0 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ \lambda_{21} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \lambda_{42} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \lambda_{63} \\ 0 & 0 & \lambda_{73} \\ 0 & 0 & \lambda_{83} \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \eta_{3i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \\ \varepsilon_{6i} \\ \varepsilon_{7i} \\ \varepsilon_{8i} \end{bmatrix}$$

$$\text{where } \text{VAR}(\varepsilon_i) = \Theta = \text{DIAG}(\theta_{11}, \theta_{22}, \dots, \theta_{88})$$

We'll talk more about scaling latent variables in SEM shortly. For now we just use scaling items.

Model Specification: Example

- ▶ The *structural model* is specified as

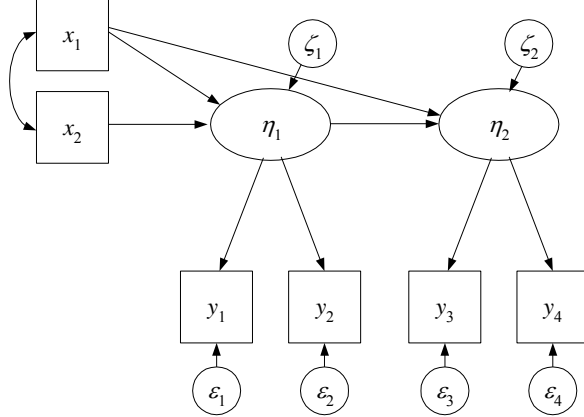
$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \eta_{3i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \beta_{21} & 0 & 0 \\ 0 & \beta_{32} & 0 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ \eta_{3i} \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \\ \zeta_{3i} \end{bmatrix}$$

where

$$\text{VAR}(\zeta_i) = \Psi = \begin{bmatrix} \psi_{11} & & \\ 0 & \psi_{22} & \\ 0 & 0 & \psi_{33} \end{bmatrix}$$

Model Specification: Example 2

- Let us now consider the example model with exogenous x 's:



Model Specification: Example 2

- Using scaling indicators to set the metric of the latent variables, the *measurement model* is

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \\ 0 \\ v_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \lambda_{21} & 0 \\ 0 & 1 \\ 0 & \lambda_{42} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \end{bmatrix}$$

where

$$\text{VAR}(\boldsymbol{\varepsilon}_i) = \boldsymbol{\Theta} = \text{DIAG}(\theta_{11}, \theta_{22}, \theta_{33}, \theta_{44})$$

Model Specification: Example 2

► The structural model is specified as

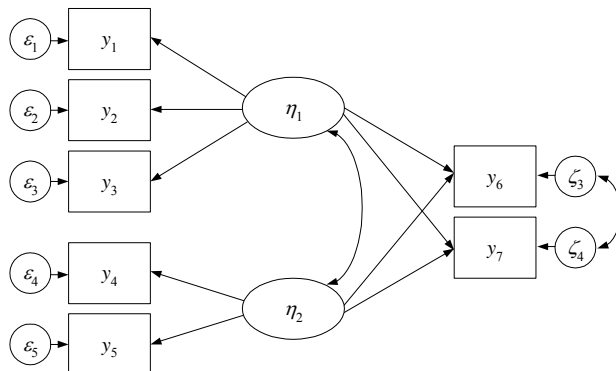
$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \beta_{21} & 0 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix} \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \end{bmatrix}$$

where

$$VAR(\zeta_i) = \Psi = \begin{bmatrix} \psi_{11} & \\ 0 & \psi_{22} \end{bmatrix}$$

Model Specification: Example 3

► Recall that our third example model includes two manifest outcomes:



Model Specification: Example 3

- Using scaling indicators to set the metric of the latent variables, the *measurement model* for the 5 indicator variables is

$$\begin{bmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \\ y_{4i} \\ y_{5i} \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \\ v_3 \\ 0 \\ v_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ 0 & 1 \\ 0 & \lambda_{52} \end{bmatrix} \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \end{bmatrix}$$

where

$$\text{VAR}(\varepsilon_i) = \Theta = \text{DIAG}(\theta_{11}, \theta_{22}, \theta_{33}, \theta_{44}, \theta_{55})$$

The variables y_6 and y_7 do not appear here as they are not indicators of a latent factor. They will appear in the structural model instead as, essentially, “manifest factors”.

Model Specification: Example 3

- The *structural model* is specified as

$$\begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ y_{6i} \\ y_{7i} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & 0 & 0 \\ \beta_{41} & \beta_{42} & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_{1i} \\ \eta_{2i} \\ y_{6i} \\ y_{7i} \end{bmatrix} + \begin{bmatrix} \zeta_{1i} \\ \zeta_{2i} \\ \zeta_{3i} \\ \zeta_{4i} \end{bmatrix}$$

where

$$\text{VAR}(\zeta_i) = \Psi = \begin{bmatrix} \psi_{11} & & & \\ \psi_{12} & \psi_{22} & & \\ 0 & 0 & \psi_{33} & \\ 0 & 0 & \psi_{43} & \psi_{44} \end{bmatrix}$$

Here we see y_6 and y_7 alongside the latent variables. The subscript numbering on the regression coefficients and disturbances is a little unfortunate because it doesn't match the numbering of the y variables. Such is life.

Model-Implied Moment Structure

- ▶ Recall that the measurement and structural models of the full SEM are written as

$$\mathbf{y}_i = \mathbf{v} + \Lambda \boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i, \quad \text{VAR}(\boldsymbol{\varepsilon}_i) = \Theta$$

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{B} \boldsymbol{\eta}_i + \Gamma \mathbf{x}_i + \boldsymbol{\zeta}_i, \quad \text{VAR}(\boldsymbol{\zeta}_i) = \Psi$$

- ▶ Together, these imply a specific moment structure. The mean structure is

$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\mu}_y(\boldsymbol{\theta}) \\ \boldsymbol{\mu}_x \end{bmatrix} = \begin{bmatrix} \mathbf{v} + \Lambda \boldsymbol{\mu}_\eta \\ \boldsymbol{\mu}_x \end{bmatrix} = \begin{bmatrix} \mathbf{v} + \Lambda (\mathbf{I} - \mathbf{B})^{-1} (\boldsymbol{\alpha} + \Gamma \boldsymbol{\mu}_x) \\ \boldsymbol{\mu}_x \end{bmatrix}$$

As in the previous models we have seen, the mean structure will often be saturated, with as many parameters estimated as there are observed means.

Model-Implied Moment Structure

- ▶ They also imply the covariance structure

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \begin{bmatrix} \boldsymbol{\Sigma}_{yy}(\boldsymbol{\theta}) & \\ \boldsymbol{\Sigma}_{xy}(\boldsymbol{\theta}) & \boldsymbol{\Sigma}_{xx} \end{bmatrix}$$

where

$$\begin{aligned} \boldsymbol{\Sigma}_{yy}(\boldsymbol{\theta}) &= \Lambda \boldsymbol{\Sigma}_{\eta\eta} \Lambda' + \Theta \\ &= \Lambda (\mathbf{I} - \mathbf{B})^{-1} (\Gamma \boldsymbol{\Sigma}_{xx} \Gamma' + \Psi) (\mathbf{I} - \mathbf{B})^{-1'} \Lambda' + \Theta \end{aligned}$$

and

$$\boldsymbol{\Sigma}_{xy}(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_{xx} \Gamma' (\mathbf{I} - \mathbf{B})^{-1'} \Lambda'$$

Notice that the expression $\boldsymbol{\Sigma}_{yy}(\boldsymbol{\theta}) = \Lambda \boldsymbol{\Sigma}_{\eta\eta} \Lambda' + \Theta$ is of the same form as the CFA model. The difference in the SEM is that we also apply a path analytic structure to $\boldsymbol{\Sigma}_{\eta\eta}$, the covariance matrix of the factors.

Model-Implied Moment Structure

- ▶ Putting all the pieces together, we have the following model-implied moment structure

$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \left[\frac{\mathbf{v} + \boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\alpha} + \boldsymbol{\Gamma}\boldsymbol{\mu}_x)}{\boldsymbol{\mu}_x} \right]$$

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \left[\begin{array}{c|c} \boldsymbol{\Lambda}(\mathbf{I} - \mathbf{B})^{-1}(\boldsymbol{\Gamma}\boldsymbol{\Sigma}_{xx}\boldsymbol{\Gamma}' + \boldsymbol{\Psi})(\mathbf{I} - \mathbf{B})^{-1'}\boldsymbol{\Lambda}' + \boldsymbol{\Theta} & \\ \hline \boldsymbol{\Sigma}_{xx}\boldsymbol{\Gamma}'(\mathbf{I} - \mathbf{B})^{-1'}\boldsymbol{\Lambda}' & \boldsymbol{\Sigma}_{xx} \end{array} \right]$$

There is a lot going on here, but the important thing is just to realize that the model implies a specific structure for the means and covariances of the data, and one of our key objectives is to see how well this structure matches the observed means and covariances.

Model Identification

- ▶ Like CFA, one requirement for model identification is that the scale of the latent variables be defined through specific restrictions in the model.
- ▶ As before, one way to set the scale of the latent factors is by fixing ν and λ to zero and one for a scaling indicator, respectively
 - ▶ This is what we did when specifying the preceding models
- ▶ Alternatively, we can set the scale of the latent variables by fixing α and ψ to zero and one, respectively.
 - ▶ Unlike CFA, this is not always equivalent to putting the factor on a standard normal scale.
 - ▶ If the factor is predicted by other variables in the model, then we are actually standardizing the factor residuals, and the total variance of the factor will exceed one

One can put the factors on a standard normal scale post-estimation by requesting the standardized solution. We will see this in our example later.

Model Identification

- ▶ Aside from assigning a metric to the latent variables, other restrictions are also necessary to identify the model.
 - ▶ e.g., common to assume simple structure and local independence of items (but not required)
 - ▶ Model identification can be determined by
 - ▶ Matrix algebra (hard)
 - ▶ t -rule (necessary but not sufficient)
 - ▶ Two-step rule (sufficient but not necessary)
-

t -Rule

- ▶ As before, the t -rule simply stipulates that there should be more observed means and unique (co)variances than estimated parameters.
- ▶ More formally, if t is the number of freely estimated parameters in θ then a necessary (but not sufficient) condition for identification is

$$t \leq (p + q)(p + q + 1)/2 + (p + q)$$

where p and q are the number of y and x variables, respectively

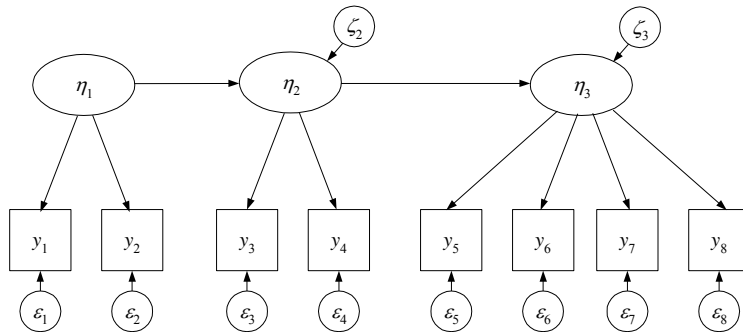
- ▶ Because μ_x and Σ_{xx} are always saturated, some software programs (e.g., Mplus) do not count the elements in these matrices as parameters to be estimated
 - ▶ In assessing the t -rule, however, one should include unique elements in μ_x and Σ_{xx} when counting the number of parameters t .
-

Two-Step Rule

- ▶ Step 1: Respecify the model as a confirmatory factor analysis with all possible associations between factors. Determine the identification of the measurement model as you would do for a CFA.
- ▶ Step 2: Treat all latent variables within the structural model as if they were observed variables. Determine the identification of the structural model as you would do for a simultaneous equation model for observed variables.
- ▶ Passing the two-step rule is sufficient to ensure model identification, but not necessary (i.e., some identified models fail the two-step rule).

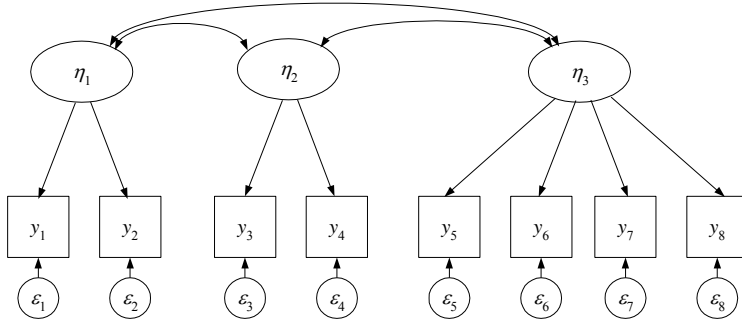
Example of Two-Step Rule

- ▶ Consider the following model



Example of Two-Step Rule

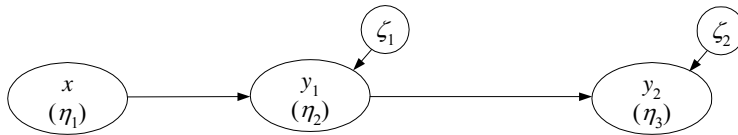
- ▶ The two-step rule involves first respecifying the model as a CFA with all possible associations between factors



- ▶ This model is identified by the two-indicator rule.

Example of Two-Step Rule

- ▶ Second, we evaluate the structural model as we would a simultaneous equation model for observed variables



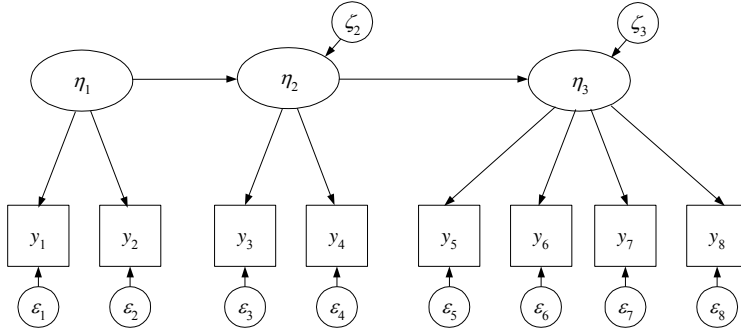
- ▶ The structural model is identified by the recursive rule (the **B** matrix can be written as a lower triangular matrix and the **Ψ** matrix is diagonal)

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ \beta_{21} & 0 \end{bmatrix} \quad \mathbf{\Psi} = \begin{bmatrix} \psi_{11} & \\ 0 & \psi_{22} \end{bmatrix}$$

Note that the first latent variable is now treated as if it were a fixed-*x* exogenous predictor, so the **B** and **Ψ** matrices are only 2 x 2 for the simultaneous equation model (whereas they are 3 x 3 for the SEM as actually fit to the data).

Example of Two-Step Rule

- ▶ Because this model satisfies the two-step rule, that is sufficient to say that the model is identified



- ▶ Empirical under-identification is still possible (e.g., if certain structural paths or factor loadings approach values of zero)

Model Estimation

- ▶ Given identification of model, we can now turn to estimation
- ▶ As with simultaneous equation models and CFA, multiple estimators exist for SEMs
 - ▶ Maximum likelihood
 - ▶ Ordinary least squares
 - ▶ Generalized least squares
 - ▶ Weighted least squares
- ▶ As before, we will concentrate on maximum likelihood
 - ▶ The only significance difference from path analysis and CFA is in the computation of the model-implied means and covariances

Model Estimation

- ▶ The normal-theory ML fitting function is again

$$-2\ell(\boldsymbol{\theta}) = \sum_{i=1}^N n_i \ln(2\pi) + \ln |\boldsymbol{\Sigma}_i(\boldsymbol{\theta})| + \left[(\mathbf{z}_i - \boldsymbol{\mu}_i(\boldsymbol{\theta}))' \boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\theta}) (\mathbf{z}_i - \boldsymbol{\mu}_i(\boldsymbol{\theta})) \right]$$

- ▶ where $\mathbf{z}'_i = [\mathbf{y}'_i \quad \mathbf{x}'_i]$, i.e., \mathbf{y}_i is stacked on \mathbf{x}_i .
- ▶ The i subscripts on n , $\boldsymbol{\Sigma}$, and $\boldsymbol{\mu}$ are there to allow for missing data.
- ▶ As before, an equivalent ML fitting function can be written in terms of sufficient summary statistics only

$$F_{ML}(\boldsymbol{\theta}) = \ln |\boldsymbol{\Sigma}(\boldsymbol{\theta})| + \text{tr}[\mathbf{S}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta})] - \ln |\mathbf{S}| - (p+q) + [\mathbf{m} - \boldsymbol{\mu}(\boldsymbol{\theta})]' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\theta}) [\mathbf{m} - \boldsymbol{\mu}(\boldsymbol{\theta})]$$

- ▶ Assumes complete data

n_i is the number of observed values in \mathbf{z}_i for case i .

Evaluation of Parameter Estimates

- ▶ Interpretation of the measurement model is like CFA, and interpretation of the structural model is like path analysis
- ▶ Can again compute standardized parameter estimates to aid evaluation.
 - ▶ For factor loadings and causal structural paths this is akin to computing standardized regression coefficients
 - ▶ For covariance parameters, rescaling to correlation metric
 - ▶ For indicator residual variances, rescaled to variance unexplained
- ▶ Can compute communality, or h^2 , for indicators
- ▶ Can compute variance explained, or R^2 , for latent or manifest outcomes

Model Respecification

- ▶ When the model fails to fit, modification indices may suggest avenues for improvement.
 - ▶ As always, one must treat these with some skepticism, as use of MIs is exploratory and model fit can be improved by capitalizing on chance.
 - ▶ MIs indicate expected improvement in fit for freeing a single parameter at a time, as opposed to larger scale model misspecifications.
-

Model Respecification

- ▶ Another strategy for locating model misfit is to isolate problems to the structural or measurement model.
 - ▶ Useful only if the structural model is not saturated.
 - ▶ Similar in concept to the two-step rule for identification.
 - ▶ Overall model χ^2 can be decomposed into additive components due to misfit of the measurement model + misfit of the structural model.
 - ▶ The component for the measurement model can be obtained by fitting a CFA respecification of the model with a saturated ψ matrix. The χ^2 obtained for this model is the misfit due to the measurement model.
 - ▶ Poor fit indicates a problem with the measurement model.
 - ▶ The difference in χ^2 between the CFA and original SEM is the misfit due to the structural model. Can evaluate by LRT, given these are nested models (Anderson & Gerbing, 1988).
 - ▶ A significant χ^2 difference test indicates a problem with the structural model
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Summary

- ▶ In sum, the specification, identification, and process of fitting and interpreting SEMs all follow straightforwardly from procedures for simultaneous equation models and CFA models.
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5.3 Example SEM

Objectives

- ▶ Provide a real-data example of the process of fitting, evaluating, re-specifying, and interpreting an SEM
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Example: Posttraumatic Growth

- ▶ Senol-Durak & Ayvasik (2010) wished to determine what factors are associated with posttraumatic growth in spouses of myocardial infarction patients.
- ▶ Sample size of 132.
- ▶ Hypotheses were that
 - ▶ Environmental Resources (ER, i.e., social support) and Individual Resources (IR) influence PostTraumatic Growth (PTG)
 - ▶ ER and IR have direct effects on PTG.
 - ▶ ER and IR also have indirect effects on PTG.
 - ▶ ER and IR influence Event-Related Factors (ERF)
 - ▶ ERF influences Cognitive Process Coping (Coping)
 - ▶ Cope influences PTG

It is worth noting that the sample size is a bit low, especially for the complexity of the fitted model, but this is not uncommon for published applications of SEM.

The full reference is:

Senol-Durak, E. & Ayvasik, H.B. (2010). Factors associated with posttraumatic growth among the spouses of myocardial infarction patients. *Journal of Health Psychology, 15*, 5–95.

Abstract from manuscript: To clarify the rationale behind Posttraumatic Growth (PTG), a model by Schaefer and Moos describes the relative contribution of environmental resources, individual resources, event related factors, cognitive processing and coping (CPC) on PTG. In the present study, this model was tested with the spouses of myocardial infarction patients with data from various hospitals in Turkey. A structural equation model revealed that neither individual nor environmental resources had indirect effects on PTG through the effect of event-related factors and CPC, while they showed direct effects on PTG. The findings were discussed in the context of the theoretical model.

Example: Multiple Indicator Latent Factors

▶ Environmental Resources

- ▶ Three indicators: Social support from family (*fa*), friends (*fr*), significant others (*si*)

▶ Individual Resources

- ▶ Five indicators: psychological hardiness indicators: commitment (*comt*), control (*con*), challenge (*cha*); self esteem scale scores (*es*), and locus of control scale scores (*lo*)

▶ Event-Related Factors

- ▶ Three indicators: subjective evaluations of prognosis (*pro*), threat to future health (*th*), and time since diagnosis (*time*)
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One could probably make a case that the indicators for Event-Related Factors should be causal indicators rather than effect indicators. The authors used a reflective measurement model for all factors, however, and we remain consistent with their analysis.

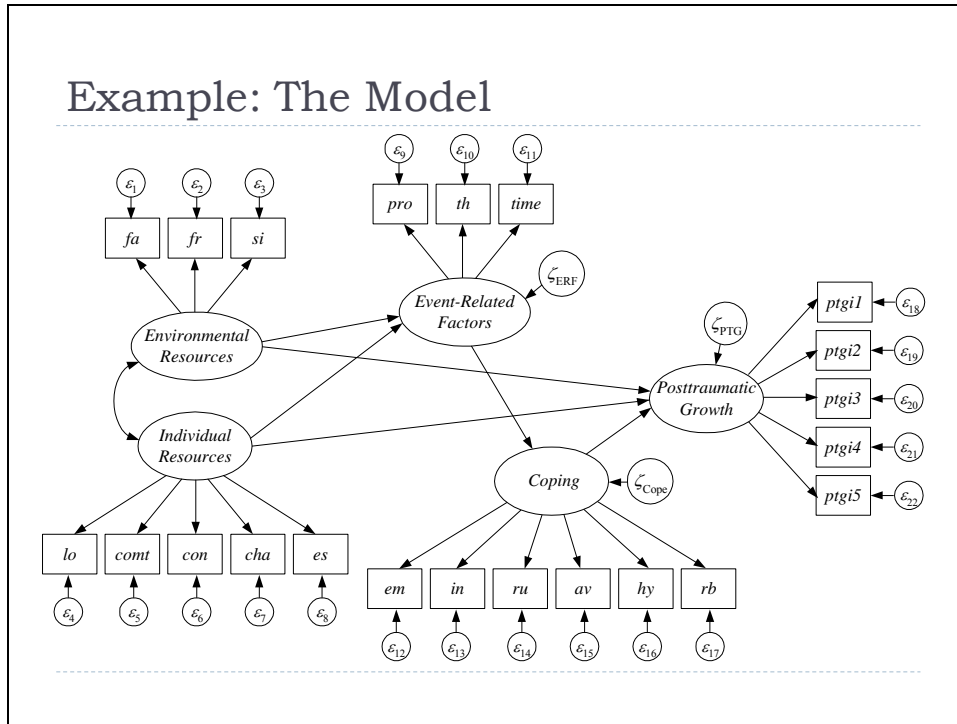
Example: Multiple Indicator Latent Factors

▶ Cognitive process coping

- ▶ Six indicators: emotion focused coping (*em*), indirect coping (*in*), rumination (*ru*), avoidance (*av*), hypervigilance (*hy*), and religious belief (*rb*)

▶ Posttraumatic growth

- ▶ Five indicators, subscale scores from posttraumatic growth inventory: improved relationship (*ptgi1*), new possibilities for one's life (*ptgi2*), greater appreciation of life (*ptgi3*), greater sense of personal strength (*ptgi4*), spiritual development (*ptgi5*).
-



Example: The Model

- ▶ Like many SEM's, the model is rather large and the matrix representation of the model is unwieldy.
- ▶ The path diagram, however, nicely conveys the proposed structure and is sufficient to infer the matrix representation of the model.
 - ▶ We thus forgo showing the matrix representation here – this is provided in the text

The measurement model is:

$$\begin{bmatrix} fa_i \\ fr_i \\ si_i \\ lo_i \\ comt_i \\ con_i \\ cha_i \\ es_i \\ pro_i \\ th_i \\ time_i \\ em_i \\ in_i \\ ru_i \\ av_i \\ hy_i \\ rb_i \\ ptgi_{1i} \\ ptgi_{2i} \\ ptgi_{3i} \\ ptgi_{4i} \\ ptgi_{5i} \end{bmatrix} = \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \\ v_{4i} \\ v_{5i} \\ v_{6i} \\ v_{7i} \\ v_{8i} \\ v_{9i} \\ v_{10i} \\ v_{11i} \\ v_{12i} \\ v_{13i} \\ v_{14i} \\ v_{15i} \\ v_{16i} \\ v_{17i} \\ v_{18i} \\ v_{19i} \\ v_{20i} \\ v_{21i} \\ v_{22i} \end{bmatrix} + \begin{bmatrix} \lambda_{11} & 0 & 0 & 0 & 0 \\ \lambda_{21} & 0 & 0 & 0 & 0 \\ \lambda_{31} & 0 & 0 & 0 & 0 \\ 0 & \lambda_{42} & 0 & 0 & 0 \\ 0 & \lambda_{52} & 0 & 0 & 0 \\ 0 & \lambda_{62} & 0 & 0 & 0 \\ 0 & \lambda_{72} & 0 & 0 & 0 \\ 0 & \lambda_{82} & 0 & 0 & 0 \\ 0 & 0 & \lambda_{93} & 0 & 0 \\ 0 & 0 & \lambda_{10,3} & 0 & 0 \\ 0 & 0 & \lambda_{11,3} & 0 & 0 \\ 0 & 0 & 0 & \lambda_{12,4} & 0 \\ 0 & 0 & 0 & \lambda_{13,4} & 0 \\ 0 & 0 & 0 & \lambda_{14,4} & 0 \\ 0 & 0 & 0 & \lambda_{15,4} & 0 \\ 0 & 0 & 0 & \lambda_{16,4} & 0 \\ 0 & 0 & 0 & \lambda_{17,4} & 0 \\ 0 & 0 & 0 & 0 & \lambda_{18,5} \\ 0 & 0 & 0 & 0 & \lambda_{19,5} \\ 0 & 0 & 0 & 0 & \lambda_{20,5} \\ 0 & 0 & 0 & 0 & \lambda_{21,5} \\ 0 & 0 & 0 & 0 & \lambda_{22,5} \end{bmatrix} \begin{bmatrix} \eta_{ERi} \\ \eta_{IRi} \\ \eta_{ERFi} \\ \eta_{COPEi} \\ \eta_{PTGi} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \\ \varepsilon_{5i} \\ \varepsilon_{6i} \\ \varepsilon_{7i} \\ \varepsilon_{8i} \\ \varepsilon_{9i} \\ \varepsilon_{10i} \\ \varepsilon_{11i} \\ \varepsilon_{12i} \\ \varepsilon_{13i} \\ \varepsilon_{14i} \\ \varepsilon_{15i} \\ \varepsilon_{16i} \\ \varepsilon_{17i} \\ \varepsilon_{18i} \\ \varepsilon_{19i} \\ \varepsilon_{20i} \\ \varepsilon_{21i} \\ \varepsilon_{22i} \end{bmatrix}$$

where $\Theta = DIAG(\theta_{11}, \theta_{22}, \dots, \theta_{22,22})$

The latent variable model is:

$$\begin{bmatrix} \eta_{ERi} \\ \eta_{IRi} \\ \eta_{ERFi} \\ \eta_{COPEi} \\ \eta_{PTGi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \beta_{31} & \beta_{32} & 0 & 0 & 0 \\ 0 & 0 & \beta_{43} & 0 & 0 \\ \beta_{51} & \beta_{52} & 0 & \beta_{54} & 0 \end{bmatrix} \begin{bmatrix} \eta_{ERi} \\ \eta_{IRi} \\ \eta_{ERFi} \\ \eta_{COPEi} \\ \eta_{PTGi} \end{bmatrix} + \begin{bmatrix} \zeta_{ERi} \\ \zeta_{IRi} \\ \zeta_{ERFi} \\ \zeta_{COPEi} \\ \zeta_{PTGi} \end{bmatrix}$$

$$\text{where } \Psi = \begin{bmatrix} 1 & & & & & \\ \psi_{21} & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \end{bmatrix}$$

Note that the means/intercepts and (residual) variances of the factors have been fixed to 0 and 1, respectively to scale the latent variables.

Example: Identification

- ▶ We shall scale the latent factors/residuals by setting their means and variances to 0 and 1, respectively
- ▶ Let us now see if the model passes the t -rule.
- ▶ The number of estimated parameters is 22 factor loadings + 22 intercepts + 22 residual variances + 1 factor correlation + 6 factor regression slopes = 73
- ▶ The number of observed variables is 22, so there are

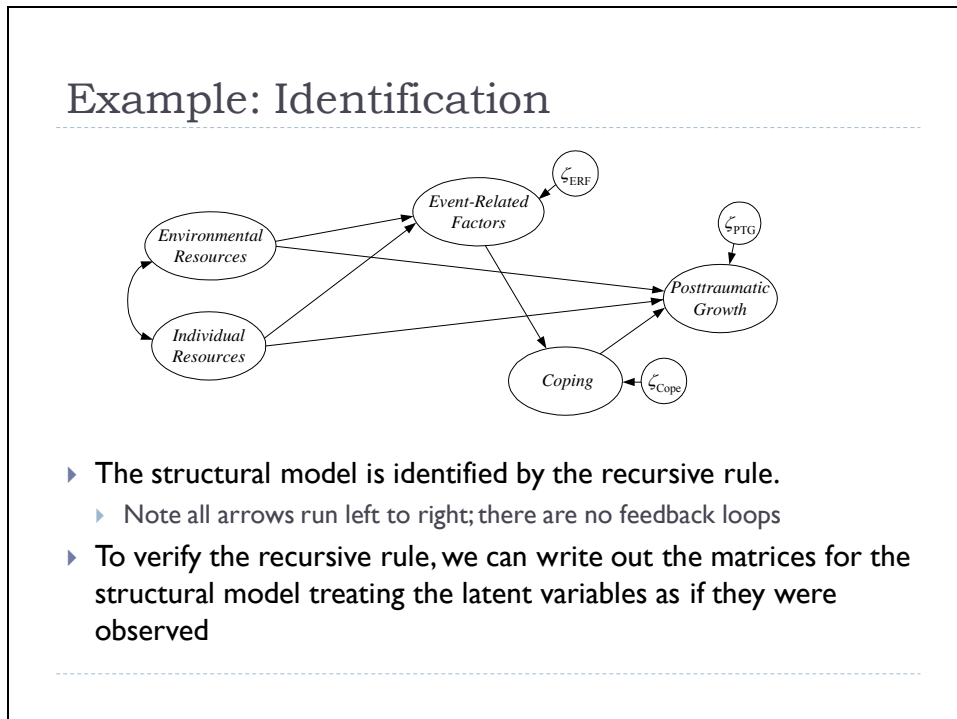
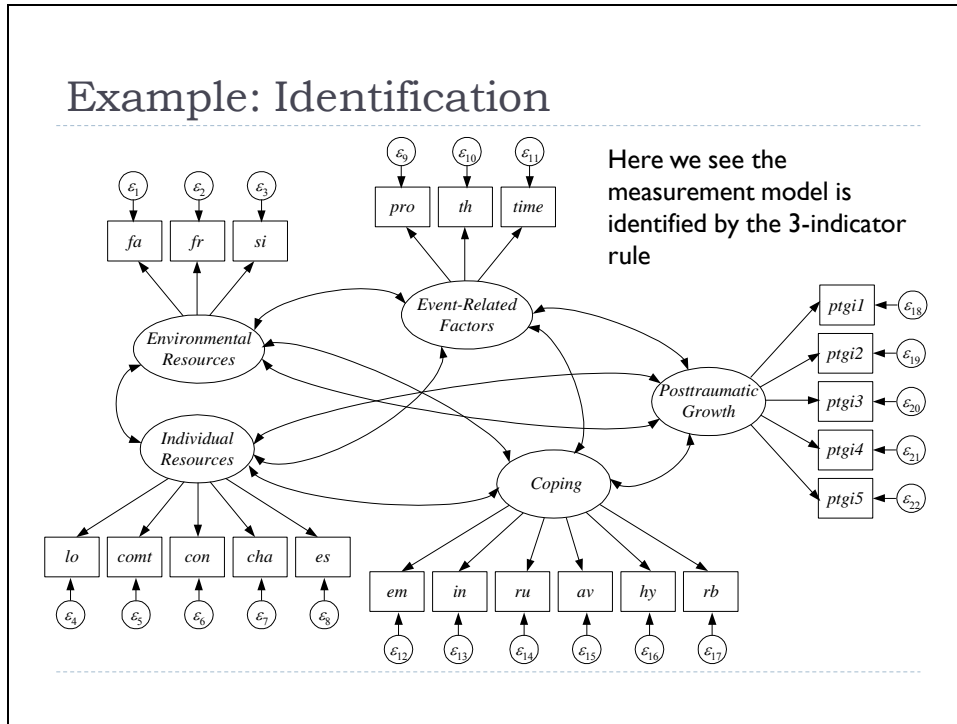
$$22(22+1)/2 + 22 = 275$$

unique observed first- and second moments from which to fit the model

- ▶ The model thus passes the t -rule and will have $275 - 73 = 202$ degrees of freedom.
-

Example: Identification

- ▶ The t -rule is necessary but not sufficient.
 - ▶ It quickly shows us that our model *may* be identified.
 - ▶ Now we can turn to the two-step rule to further evaluate the identification of the model.
 - ▶ Recall that the first step is to reparameterize the model as a confirmatory factor model with all possible associations among the factors to verify identification of the measurement model.
 - ▶ The second step is to examine the structural model as if it were a structural model for observed variables to verify its identification.
-



No causal feedback loops or correlated disturbances are present here.

Example: Identification

- ▶ If the latent variables were observed, the simultaneous equation model would be

$$\begin{bmatrix} ERF_i \\ Coping_i \\ PTG_i \end{bmatrix} = \begin{bmatrix} \alpha_{ERF} \\ \alpha_{COPE} \\ \alpha_{PTG} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \beta_{21} & 0 & 0 \\ 0 & \beta_{32} & 0 \end{bmatrix} \begin{bmatrix} ERF_i \\ Coping_i \\ PTG_i \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ 0 & 0 \\ \gamma_{31} & \gamma_{32} \end{bmatrix} \begin{bmatrix} ER_i \\ IR_i \end{bmatrix} + \begin{bmatrix} \zeta_{ERFi} \\ \zeta_{COPEi} \\ \zeta_{PTGi} \end{bmatrix}$$

with

$$\Psi = \begin{bmatrix} \psi_{ERF} & & \\ 0 & \psi_{COPE} & \\ 0 & 0 & \psi_{PTG} \end{bmatrix}$$

- ▶ The recursive model is satisfied because the **B** matrix is lower triangular and the ψ matrix is diagonal.

Note that the Environmental and Individual Resources factors are treated as if they were exogenous x variables in this step.

Example: Fitting the Model

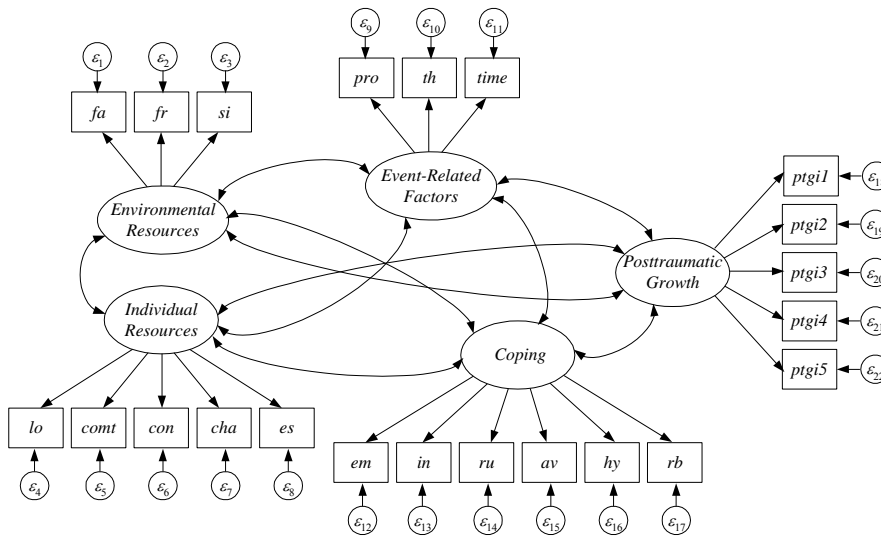
- ▶ Since the model is identified, we can proceed to fit it to the data
 - ▶ Caution: $N = 132$ is rather low for a model of this complexity.
- ▶ Senol-Durak & Ayvasik (2010) provided their correlation matrix, standard deviations, and means. We can therefore replicate their analysis via the summary data version of ML estimation.
- ▶ We obtain the following fit measures
 - ▶ $\chi^2(202) = 350.00, p < .0001$
 - ▶ CFI = .848; TLI = .826
 - ▶ RMSEA = .075; CI90 = (.061, .087)
 - ▶ SRMR = .101
- ▶ These fit statistics suggest mediocre fit of the model to the data.

Our fit measures do not perfectly reproduce the results reported in Senol-Durak & Ayvasik (2010) but this may be due to rounding error in the correlation matrix.

Example: Model Respecification

- ▶ The largest MIs for the model are associated with residual covariances and cross-loadings. This suggests a problem with the measurement model.
- ▶ To more formally locate the source of misfit, we can respecify the model as a CFA model.
 - ▶ Poor absolute fit indicative of misfit of measurement model.
 - ▶ Significant chi-square difference test with original SEM indicative of misfit of structural model.

Fitting as CFA Model



Fitting as CFA Model

- ▶ Fitting the CFA model to test the measurement model, we obtain:
 - ▶ $\chi^2(199) = 348.64, p < .0001$
 - ▶ CFI = .846; TLI = .822
 - ▶ RMSEA = .075; CI90 = (.062, .088)
 - ▶ SRMR = .099
 - ▶ Conducting the chi-square difference test between the original SEM and the CFA to test the structural model, we obtain:
 - ▶ $\Delta\chi^2(3) = 350.00 - 348.64 = 1.36, p = .71$.
 - ▶ In line with the MIs, these results show that the mediocre fit is a reflection of problems with the measurement model.
-

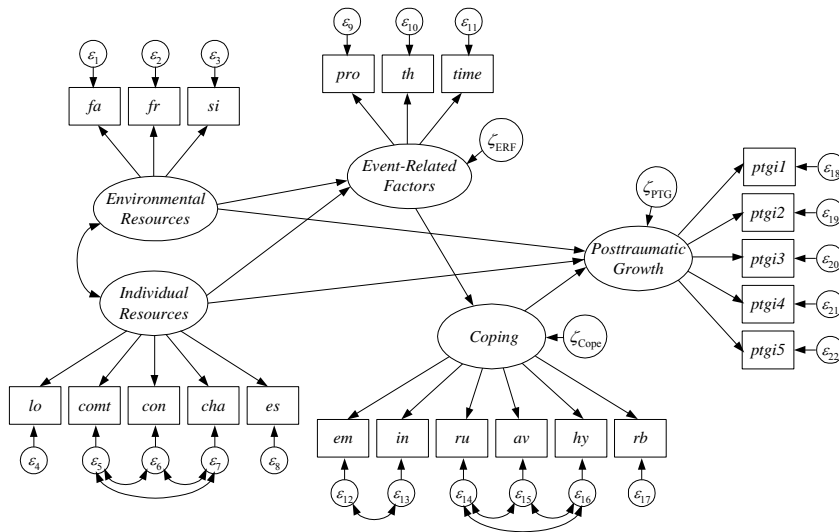
Respecifying the Model

- ▶ Closer examination of the indicators suggests one possible respecification of the model.
 - ▶ For some factors, subsets of indicators are composed of subscales from a single scale
 - ▶ Individual Resources factor includes three Psychological Hardiness subscale scores as indicators (*comt*, *con*, *cha*), but also two indicators from independent scales (*lo* and *es*).
 - ▶ Coping factor includes two indicators from the Ways of Coping Inventory (*em* and *in*), three indicators from the Impact of Event Scale (*ru*, *av*, & *hy*), and a religious beliefs score from another scale (*rb*).
-

Respecifying the Model

- ▶ We will use a correlated uniqueness model to account for possible scale factors
 - ▶ Indicators from the same scale likely to be more highly related than indicators from different scales, above and beyond common factor
- ▶ Respecifying the model in this way is theoretically motivated but is also consistent with the MIs
 - ▶ Four of the five highest MIs are for residual covariances between subscale scores from the same scale:
 - ▶ $\theta_{hy,ru}$ **MI = 32.258**
 - ▶ $\theta_{cha,comt}$ **MI = 27.845**
 - ▶ $\theta_{in,em}$ **MI = 21.340**
 - ▶ $\theta_{in,fa}$ **MI = 14.250**
 - ▶ $\theta_{cha,com}$ **MI = 12.345**

Revised Model

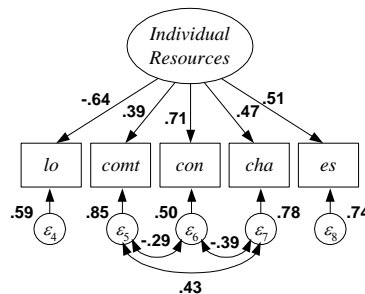


Revised Model

- ▶ Fit statistics for the correlated uniqueness model:
 - ▶ $\chi^2(195) = 270.28, p = .0003$
 - ▶ CFI = .923; TLI = .908
 - ▶ RMSEA = .054; CI90 = (.037, .069)
 - ▶ SRMR = .090
- ▶ CU model fits significantly better than the original model:
 - ▶ $\Delta\chi^2(7) = 350.00 - 270.28 = 79.72, p < .0001$.
- ▶ Overall fit is borderline, especially given low power with low N
 - ▶ Some large MIs still, but suggested paths not consistent with theory
 - ▶ Especially at low N , suggested paths may reflect only chance fluctuations in data that would not replicate in another sample
- ▶ We will retain the revised, correlated uniqueness model and proceed to interpret the estimates.

Interpretation

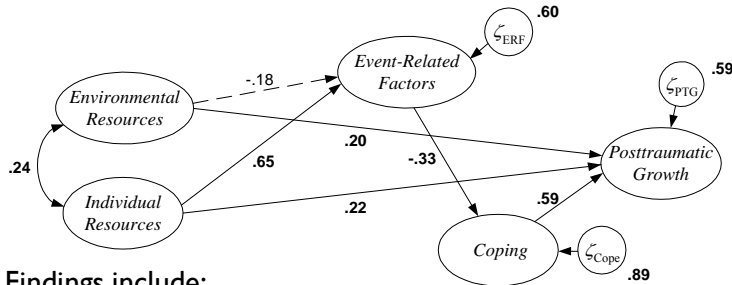
- ▶ Interpretation of the measurement model is done similarly to CFA. We will thus consider just the IR factor here as an example
 - ▶ Note that locus of control is a negative indicator of individual resources (but not clear from article how this was scored)
 - ▶ Control subscale of Psychological Hardiness Scale is negatively correlated with commitment and challenge subscales
 - ▶ unusual for a CU model & inconsistent with notion that residuals correlate due to a common "scale" factor



All reported values are standardized estimates. Although we consider only the Individual Resources factor here, the interpretation of the other factors would proceed similarly.

Interpretation

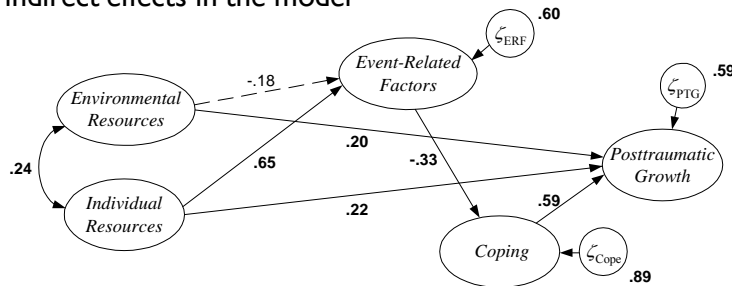
- ▶ The structural is depicted with standardized estimates below:



- ▶ Findings include:
 - ▶ Environmental resources (social support) and Individual Resources positively influence posttraumatic growth
 - ▶ Coping positively influences posttraumatic growth
 - ▶ Positive event related factors (better prognosis, less threat, more time from event) predict less coping

Total, Direct, and Indirect Effects

- ▶ Note, however, that there is a complex pattern of direct and indirect effects in the model



- ▶ To fully interpret the model, we need to evaluate total, direct and indirect effects
 - ▶ The computation of these effects and their standard errors mimics the procedures discussed earlier for path analysis

Effect Decomposition

- ▶ Let's first consider the effect of Event-Related Factors on Post-Traumatic Growth
 - ▶ The effects of Event-Related Factors on Post-Traumatic Growth:
 - ▶ Total: $-.20, p < .05$
 - ▶ Direct: 0
 - ▶ Indirect: $-.20, p < .05$
 - ▶ Interpretation: More positive event-related factors reduce the need for Coping. Since Coping positively impacts Post-Traumatic Growth, higher Event-Related Factors lead to less opportunity for Post-Traumatic Growth
 - ▶ Note that the direct effect of Event-Related Factors on Post-Traumatic Growth was constrained to be zero, so the total effect equals the indirect effect in this case.
-

Effect Decomposition

- ▶ Let us now consider the effects of our distal predictors, Environmental Resources and Individual Resources
 - ▶ The effects of Environmental Resources on Post-Traumatic Growth:
 - ▶ Total: $.24, p < .05$
 - ▶ Direct: $.20, p < .05$
 - ▶ Indirect: $.04, ns$
 - ▶ Environmental Resources do not significantly impact Event-Related Factors, and this is reflected in the non-significant indirect effect of Environmental Resources on Post-Traumatic Growth through Event-Related Factors and Coping.
 - ▶ Environmental Resources do, however, have a significantly positive direct effect on Post-Traumatic Growth.
-

Effect Decomposition

- ▶ Last we can consider the effect decomposition for Individual Resources
 - ▶ The effects of Individual Resources on Post-Traumatic Growth:
 - ▶ Total: .09, *ns*
 - ▶ Direct: .22, $p < .05$
 - ▶ Indirect: -.13, $p = .05$
 - ▶ There are countervailing indirect and direct effects that result in a non-significant total effect.
 - ▶ Higher Individual Resources predict more positive Event-Related Factors (which predict less Coping and, in turn, less opportunity for Post-Traumatic Growth), leading to the negative indirect effect and null total effect.
-

Summary

- ▶ The procedures we followed in fitting the SEM are very similar to those used in path analysis and CFA
 - ▶ Like in path analysis, our models may include complex causal chains and mediation effects, but now including latent predictors, mediators and outcomes.
 - ▶ The computation, testing, and interpretation of total, direct and indirect effects is similar to path analysis, and can aid in model interpretation.
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5.4 Equivalent Models

Objectives

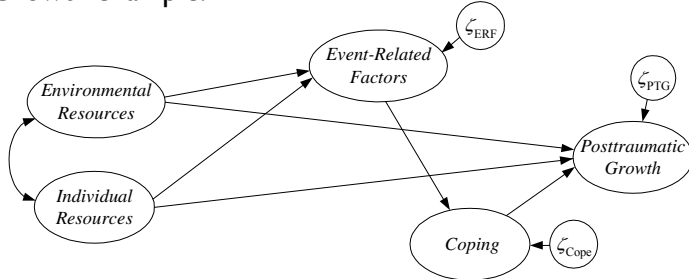
- ▶ Revisit the issue of equivalent models in the SEM context
-

Equivalent Models

- ▶ When one tests an SEM, generally one considers this a test of the particular SEM that has been specified.
 - ▶ But in fact it is a test of the class of all equivalent models to the specified SEM.
 - ▶ Some equivalent models, despite having equal fit, may generate quite different substantive conclusions, so it is a good idea to consider what alternative model structures cannot be differentiated from the hypothesized one.
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Example: Post-Traumatic Growth

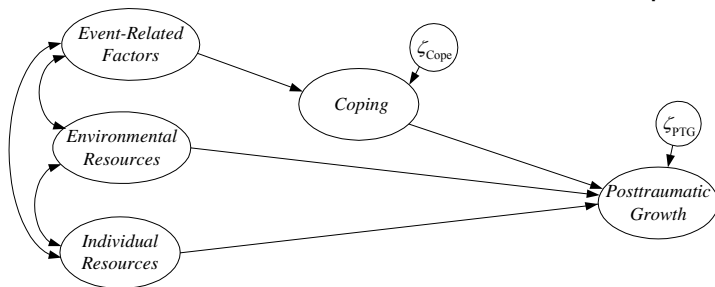
- ▶ Consider the structural model that we fit in the Post-Traumatic Growth example:



- ▶ In this model, Event-Related Factors constitute a partial mediator of the effects of Environmental Resources and Individual Resources, and Coping totally mediates the effect of Event-Related Factors

Example: Post-Traumatic Growth

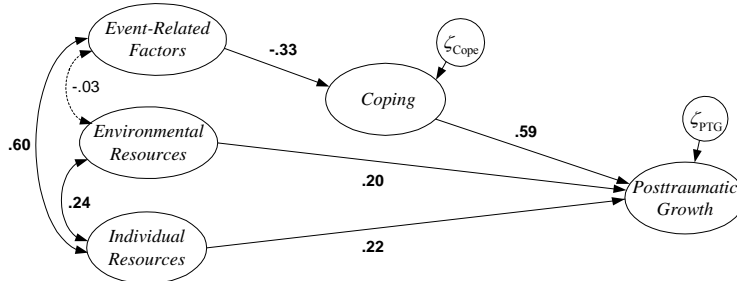
- ▶ Holding the measurement model constant, this is an equivalent structural model for the Post-Traumatic Growth example:



- ▶ This model is substantively quite different from the original, because Event-Related Factors and Coping are no longer construed as partial mediators of the effects of Environmental Resources and Individual Resources

Example: Post-Traumatic Growth

- ▶ Examining the standardized estimates, this equivalent model seems in some ways more sensible



- ▶ More positive Event-Related Factors lead to less need for coping and in turn less opportunity for post-traumatic growth
- ▶ Environmental and Individual Resources promote post-traumatic growth

Summary

- ▶ Many other equivalent models may exist that provide precisely the same fit to the data as the one initially motivated by theory
- ▶ Thus the test of theory may not be as strong as initially supposed (not just testing original model, but also all equivalent models)
- ▶ In some cases, some equivalent models may be equally or more defensible / interpretable as the original model

Chapter Summary

- ▶ Structural equation models with latent variables blend together the core features of path analysis and confirmatory factor analysis
 - ▶ SEMs are composed of two parts
 - ▶ Measurement Model (like CFA)
 - ▶ Structural Model (like path analysis)
 - ▶ Fitting and interpreting SEMs involves similar procedures to path analysis and CFA
 - ▶ Identification, estimation, evaluation, re-specification, interpretation
 - ▶ Interpretation aided by computation of total, direct, and indirect effects
 - ▶ Relative to path analysis, SEM with latent variables provides the key advantage that the coefficients in the structural model are unbiased by measurement error
-